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1989 J. Phys. A: Math. Gen. 22 731

(http://iopscience.iop.org/0305-4470/22/6/023)

J. Phys. A: Math. Gen. **22** (1989) 731-750. Printed in the UK

Finite-size scaling spectra in the six-states quantum chains

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Received 17 February 1988, in final form 26 September 1988

Abstract. The finite-size scaling spectra of the six-states quantum chains with D_6 symmetry are studied numerically in the massless phase. At least three values of the central charge c of the Virasoro algebra are found: $c=1$, 1.25 and 1.3. On the self-dual line, one finds, in the $c=1$ region, $N=2$ superconformal invariance. In the $c=1.25$ region there is a line where $N=1$ superconformal and Zamolodchikov–Fateev symmetry is observed. The occurrence of higher symmetries is also discussed.

1. Introduction

The n-states models have a long history in statistical mechanics (José *et al* 1977, Kadanoff 1979, Fradkin and Kadanoff 1980, Nienhuis 1984). Their transfer matrices are related to one-dimensional quantum chains. In this paper we study the critical behaviour of the self-dual six-states quantum chain with next-neighbour interaction, for N sites defined by the D_6-symmetric Hamiltonian (von Gehlen and Rittenberg 1986b)

$$H=-\frac{1}{\xi}\sum_{i=1}^{N}[\sigma_i+\sigma_i^5+\varepsilon(\sigma_i^2+\sigma_i^4)+\delta\sigma_i^3$$
$$+\lambda(\Gamma_i\Gamma_{i+1}^5+\Gamma_i^5\Gamma_{i+1}+\varepsilon(\Gamma_i^2\Gamma_{i+1}^4+\Gamma_i^4\Gamma_{i+1}^2)+\delta\Gamma_i^3\Gamma_{i+1}^3)] \qquad (1.1)$$

with the boundary condition

$$(\Gamma_{N+1})^m=B^{mn}\Gamma_1^n. \qquad (1.2)$$

The matrix B^{mn}, specifying the boundary condition, and the normalisation ξ, which fixes the Euclidean timescale, will be discussed below. λ plays the part of the inverse temperature, ε and δ are coupling constants, σ and Γ are given by the matrices

$$\sigma=\begin{pmatrix}1&0&0&0&0&0\\0&\omega&0&0&0&0\\0&0&\omega^2&0&0&0\\0&0&0&\omega^3&0&0\\0&0&0&0&\omega^4&0\\0&0&0&0&0&\omega^5\end{pmatrix} \qquad \Gamma=\begin{pmatrix}0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix}$$

$$\omega=\exp(2\pi i/6).$$

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For specific values of the coupling constants, on which depends the global symmetry of H (Badke *et al* 1985) one obtains different known models, e.g. for $\varepsilon=\delta=1$ the six-states Potts model or for $\varepsilon=\delta=0$ the vector Potts models. It turns out that the coupling constants determine the critical behaviour of the chain.

In the past five years the understanding of phase transitions of two-dimensional systems has developed enormously. The concept of universality has changed and become more precise in the light of modular invariance (an example with be given here). All the unitary conformal field theories, underlying certain models in statistical mechanics with central charge $c<1$ of the Virasoro algebra, can be classified (Belavin *et al* 1984, Friedan *et al* 1984, Cardy 1986, Cappelli *et al* 1987).

Subsequently much attention has been paid to the classification of two-dimensional conformally invariant theories with central charge of the Virasoro algebra $c \geqslant 1$. Higher symmetries beyond conformal invariance restrict the central charge of unitary theories to certain quantised values and allow the calculation of all possible anomalous dimensions and multipoint correlation functions. Thus the investigation of infinite Lie algebras in two dimensions turns out to be crucial for the complete understanding of conformal field theories.

However, following this concept, one does not know in which particular physical system a certain symmetry is realised or whether it is realised at all. One has a theory, but no 'experiment' to which it may be applied. In this paper we choose the by now well established inverse philosophy and start with a particular model, the six-states quantum chain (1.1), by investigating the operator content numerically under application of finite-size scaling methods. In a second step we examine which infinite Lie algebras determine the spectrum we observed numerically. As any kind of 'experiment' this method provides the possibility of discovering new interesting phenomena not yet taken into consideration from the purely theoretical point of view.

Indeed, while in earlier days the six-states model was thought to be only Gauss type (Fradkin and Kadanoff 1980), our observations show that there are different types of second-order phase transitions. We find a system with varying central charge c and a curve in the space of coupling constants with $c=\frac{5}{4}$ where the model exhibits $N=1$ supersymmetry (SUSY) and Zamolodchikov–Fateev (ZF) symmetry. We will clarify the connection between these two symmetries in terms of character identities. Furthermore we will observe $N=1$ and $N=2$ SUSY in a region with $c=1$. Here a more thorough investigation of the operator content will lead to representations of a new algebra which contains the $N=2$ superconformal algebra as a subalgebra.

The paper is organised as follows. In §§ 2 and 3 we discuss the phase diagram of the Hamiltonian (1.1) and the boundary conditions and symmetries of the system. Section 4 briefly reviews some known consequences from conformal invariance, its $N=1$ and $N=2$ supersymmetric extensions and from ZF symmetry. Furthermore the character identities of the $N=1$ SUSY and ZF symmetry at $c=\frac{5}{4}$ will be given. In § 5 we present and discuss the full operator content of (1.1) with special values of the coupling constants for free and all toroidal boundary conditions and discuss the results. Finally in § 6 our results will be summarised.

2. Phase diagram of the six-states quantum chain

The aim of this paper is a more thorough examination of the model at the critical temperature $\lambda=1$ in the region of ferromagnetic interaction which is defined by

$$\varepsilon>-1 \qquad \varepsilon>-\tfrac{1}{3}(1+2\delta) \qquad \delta>-2. \tag{2.1}$$

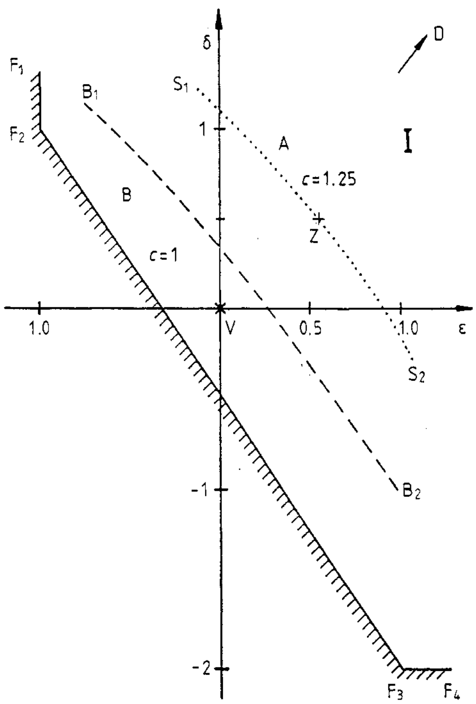

Figure 1. Parameter plane of the Hamiltonian (1.1). Line $F_1F_2F_3F_4$ marks the border of the ferromagnetic domain of interaction (I). A region of central charge $c = 1.25$ is marked by A. Along the dotted line S_1S_2, including the ZF point Z, we find $N = 1$ superconformal and ZF invariance. In region B, bounded by B_1B_2, with the vector Potts point V the model has $c = 1$ and exhibits $N = 2$ supersymmetry. The arrow points to D, where the system decouples into the Ising and three-states Potts model.

Figure 1 shows the corresponding part of the phase diagram in which the ferromagnetic domain is limited by the line $F_1F_2F_3F_4$.

The parameters ε and δ determine the critical properties of the Hamiltonian (1.1). Already, numerical examinations by von Gehlen and Rittenberg (1986b, 1987) have come to the result that, at the critical temperature $\lambda = 1$ for three special values of the coupling constants ($\delta = 1$, $\varepsilon_1 = 0$, $\varepsilon_2 = \frac{1}{3}$, $\varepsilon_3 = \frac{3}{5}$), the central charge is $c = \frac{5}{4}$. Moreover, for $\delta = 1$, $\varepsilon = 0$ there appear anomalous dimensions, which are near to those given by the highest weight representations of the $N = 1$ super-Virasoro algebra (see below).

On the other hand Zamolodchikov and Fateev (1985) constructed two-dimensional conformal field theories, which are self-dual and Z_n symmetric, and where the central charge $c = \frac{5}{4}$ if $n = 6$. The anomalous dimensions found by von Gehlen and Rittenberg are predicted by this theory, though for different values of the coupling constants ($\varepsilon = 1/\sqrt{3}$, $\delta = \frac{1}{2}$) (Alcaraz 1987a, b). At this point (Z in the plane of parameters: figure 1) Alcaraz computed numerically $c = \frac{5}{4}$ and also some of the anomalous dimensions observed by von Gehlen and Rittenberg.

The connection between these results has already been clarified (Schütz 1987). The spectrum of the Hamiltonian (2.1) for free boundary conditions on the curve S_1S_2 (figure 1) is $N = 1$ supersymmetric. Here the central charge and the critical exponents remain constant. The curve contains the ZF point Z, runs close past $\varepsilon = 0$, $\delta = 1$ (see above) and is embedded in a region A with $c = \frac{5}{4}$. However, neither the operator content for all toroidal boundary conditions nor the relation between $N = 1$ SUSY and ZF symmetry have been studied yet. This will be done in §§ 3 and 4. The critical exponents of the order parameters and the thermal exponent on this curve are the same as those of some critical points in the series of RSOS models (labelled by an integer $p>1$) (Andrews *et al* 1984), which remains true for the complete series of Z_n models of Zamolodchikov and Fateev with $n=p$. Andrews *et al* have been able to solve the RSOS models exactly on a two-dimensional subspace in a larger space of parameters. For each model they found two distinct manifolds, each of them being divided into two phases by a line of critical points. So there are four different regimes I, II, III and IV with critical lines A between I and II and B between III and IV respectively. The series of lines B describing a second-order phase transition from a N- to a $(N-1)$-phase coexistence have been identified with the series of unitary conformal field theories with $c<1$ found by Friedan *et al* (1984) (see also Huse 1984). As mentioned above, along the other line A the exponents of the $(p-1)$-order parameters and the thermal exponent are that of the Z_n theories of ZF, which are measured for the six-states case along line S_1S_2 (figure 1) with $c=\frac{5}{4}$. This multicritical transition is the continuous melting of a so-called $p\times 1$ commensurate phase (Huse 1984). On the transition line the model has a Z_n symmetry: the line separates an area of N-phase coexistence from a disordered phase. For a complete description of the model see Andrews *et al* (1984) and Huse (1984).

Moreover a region B with $c=1$ has been found (von Gehlen *et al* 1988). It is limited by the line B_1B_2 and contains the Z_6 vector Potts point V ($\varepsilon=\delta=0$). In that paper the operator content of all Z_n theories ($n>4$) in the $c=1$ region is conjectured and compared with finite-size scaling estimates. These Z_n models show a critical fan, i.e. a massless phase in an interval $1/\lambda_{\max}\leqslant 1\leqslant\lambda_{\max}$. The critical exponents are constant with respect to the coupling constants but are functions of λ. As the value of the central charge $c=1$ suggests, they are given by the Gauss model (Di Francesco *et al* 1987 and references therein):

$$\Delta=(M\pm gN)^2/4ng \tag{2.2}$$

where g is a monotonic function of λ such that

$$g(1/\lambda_{\max})=4/n \qquad g(1)=1 \qquad g(\lambda_{\max})=n/4. \tag{2.3}$$

The form of this function depends on the coupling constants (von Gehlen *et al* 1988). As an example, consider the leading magnetic exponents x_Q in the charge sector Q which turn out to be

$$x_Q=x_{n-Q}=2\Delta_Q=Q^2/2ng \qquad Q=1,2,\ldots,[n/2]. \tag{2.4}$$

Thus

$$2Q^2/n^2\leqslant x_Q\leqslant Q^2/8 \tag{2.5}$$

which recovers a known result (Elizur *et al* 1979, Cardy 1978). In addition to the part of the spectrum described by the Gauss model there are sectors with

$$\Delta=\tfrac{1}{16}+m \tag{2.6}$$

where m is integer or half-integer. This corresponds to an irreducible representation of the twisted U(1) Kac–Moody algebra (von Gehlen *et al* 1988).

Here we will show that, at $\lambda=1$, the six-states model has $N=2$ SUSY and a new symmetry which contains the $N=2$ superconformal algebra as a subalgebra.

Choosing the coupling constants as $\varepsilon = 2u/3\sqrt{3}$, $\delta = u/2$, one reaches in the limit $u \to \infty$ the point D where the system decouples into the Ising and three-states Potts model with central charge $c = \frac{1}{2} + \frac{4}{5} = 1.3$. Since the operator content of these models is known, it is trivial to calculate it for this choice of coupling constants.

Between these special values of the coupling constants the central charge as well as the critical exponents change. However, there is no marginal operator which can explain this phenomenon. That means that it is not possible to formulate a field theory which describes the system in the whole ferromagnetic region of interaction. Additionally the question arises of how can the central charge change in dependence on the coupling. The crossover from $c = 1$ to $c = \frac{5}{4}$ is going to be discussed in a separate paper. In this paper we will discuss in detail the infinite-dimensional symmetries of the Hamiltonian (1.1) in the large-N limit at the critical temperature $\lambda = 1$ in the region B and along curve S_1S_2 and present the spectrum for free and all toroidal boundary conditions.

3. Boundary conditions and symmetries

The Hamiltonian H with boundary condition B (2.2) is now called H^B. Due to the D_6 symmetry B characterises one of the twelve matrices:

$$B = \Sigma^{\tilde{Q}} C^k \qquad \tilde{Q} = 0, \ldots, 5 \qquad k = 0, 1 \tag{3.1}$$

with

$$\Sigma = \omega^m \delta_{m,n} \qquad C = \delta_{6-m,n} \qquad m, n = 1, \ldots, 5.$$

The $\Sigma^{\tilde{Q}}$ form the cyclic group Z_6 and C is the charge conjugation matrix. Together they form a reducible representation of the dihedral group D_6. The global symmetry of H depends on B; as is easily seen, H is invariant under the transformation A

$$(\Gamma^m)' = A^{mn} \Gamma^n \tag{3.2}$$

if A commutes with B. Here A again is one of the twelve matrices above. The spectrum of H depends only on the conjugacy class of D_6 that B belongs to. In table 1 the symmetry group $U \subset D_6$ built by the matrices A for each boundary condition is shown.

According to the irreducible representations (IR) of the symmetry U the Hamiltonian splits into block matrices H^B_U, called sectors (table 1).

Table 1. Possible boundary conditions and corresponding symmetries of the Hamiltonian defined by (1.1). Column 3 gives the notation of the possible sectors. In rows 1-3, Q denotes the Z_6 charge and $\pm$ the eigenvalue of charge conjugation. In row 4 the eigenvalues of Σ^3 and of B are marked by $\pm$.

Boundary condition		Symmetry	Spectra
0		D_6	$H^F_{Q,\pm}$ $Q = 0, 3$; H^F_Q $Q = 1, 2, 4, 5$
$\Sigma^{\tilde{Q}}$	$\tilde{Q} = 0, 3$	D_6	$H^{\tilde{Q}}_{Q,\pm}$ $Q = 0, 3$; $H^{\tilde{Q}}_Q$ $Q = 1, 2, 4, 5$
$\Sigma^{\tilde{Q}}$	$\tilde{Q} = 1, 2, 4, 5$	Z_6	$H^{\tilde{Q}}_Q$ $Q = 0, \ldots, 5$
$\Sigma^{\tilde{q}} C$	$\tilde{q} = 0, \ldots, 5$	$Z_2 \otimes Z_2$	$H^{\tilde{q}C}_{\pm,\pm}$

If $B=0$ (free boundary condition), $B=1$ (periodic) or $B=\Sigma^3$ (antiperiodic) we find D_6 as symmetry. The Z_6 subgroup permits a prediagonalisation into six charge sectors with eigenvalues $Q=0,\ldots,5$ of the charge operator $\hat{Q}$:

$$\hat{Q}=\sum_{i=1}^{N} q_i \qquad (\mathrm{mod}\ 6) \tag{3.3}$$

with $q=(i-1)\delta_{i,j}$, $i=1,\ldots,6$. D_6 has two irreducible two-dimensional representations D_Q, $Q=1,2$:

$$D_Q(\Sigma^l)=\begin{pmatrix}\omega^{Ql} & 0\\ 0 & \omega^{-Ql}\end{pmatrix} \qquad D_Q(C)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{3.4}$$

They mix sectors 1 and 5 as well as 2 and 4, so that

$$H_1=H_5 \qquad H_2=H_4. \tag{3.5}$$

Furthermore there are four one-dimensional representations $D_{Q,\pm}$, $Q=0,3$,

$$D_{Q,\pm}(\Sigma^l)=(-1)^{Ql} \qquad D_{Q,\pm}(C)=\pm 1 \tag{3.6}$$

which split the charge sectors 0 and 3 into two subsectors with positive and negative charge conjugation.

If $B=\Sigma^{\tilde{Q}}$, $\tilde{Q}=1,2,4,5$ (cyclic boundary conditions), the symmetry is Z_6 and only a prediagonalisation into the six charge sectors is possible.

Finally, for $B=\Sigma^{\tilde{q}}C$, $\tilde{q}=0,\ldots,5$, the symmetry remains $Z_2\otimes Z_2$, as B commutes only with itself and with Σ^3. It follows, because of $B^2=1$, that even and odd Z_6 charges ($\Sigma^3=\pm 1$) with B eigenvalues ± 1 are split into sectors. Since $H^{0C}=H^{2C}=H^{4C}$ and $H^{1C}=H^{3C}=H^{5C}$ we consider only H^{0C} and H^{3C} where $U=\{1,\Sigma^3,C,\Sigma^3C\}$ with sectors $H^{0C}_{\Sigma^3,C}$ and $H^{3C}_{\Sigma^3,C}$ respectively.

Fortunately not all these matrices are independent from each other: because of self-duality and invariance under charge conjugation only the following 25 (instead of 72) sectors are different at $\lambda=1$:

$$\begin{array}{llll}
H^F_{0,\pm} & H^F_1=H^F_5 & H^F_2=H^F_4 & H^F_{3,\pm}\\
H^0_{0,\pm} & H^0_{3,\pm}=H^3_{0,\pm} & H^3_{3,\pm} & \\
\multicolumn{2}{l}{H^0_1=H^0_5=H^5_0=H^1_0} & \multicolumn{2}{l}{H^0_2=H^0_4=H^4_0=H^2_0}\\
\multicolumn{2}{l}{H^3_1=H^3_5=H^5_3=H^1_3} & \multicolumn{2}{l}{H^3_2=H^3_4=H^4_3=H^2_3}\\
\multicolumn{2}{l}{H^1_1=H^1_5=H^5_1=H^5_5} & \multicolumn{2}{l}{H^2_2=H^2_4=H^4_2=H^4_4}\\
\multicolumn{4}{l}{H^1_2=H^1_4=H^4_1=H^2_1=H^2_5=H^4_5=H^5_4=H^5_2}\\
H^{0C}_{+,\pm} & H^{0C}_{-,\pm}=H^{3C}_{+,\pm} & H^{3C}_{-,\pm}. &
\end{array} \tag{3.7}$$

In addition H is parity and translationally (P) invariant. Thus we finally obtain matrices $H^{(\pm)}_U$ for free and $H^B_U(P)$ for toroidal boundary conditions. The eigenvalues which belong to them are called $E_k(N)$ or $E_k(P,N)$ respectively; k means a certain energy level, while N denotes the number of the sites.

4. Supersymmetry and Zamolodchikov–Fateev symmetry

In order to settle the connection between the eigenvalues of H and the representations of the Virasoro algebra we consider, first for free boundary conditions, the quantities

(Cardy 1984, 1986, von Gehlen and Rittenberg 1986a)

$$\mathscr{E}_k = \lim_{N\to\infty} (N/\pi)(E_k(N) - E_F) \tag{4.1}$$

where $E_F(N)$ is the lowest eigenvalue in spectrum $H_{0,+}^{(+)}(N)$, which means the ground state of H^F for N sites. The $\mathscr{E}_k$ define the finite-size spectrum of H^F. As a consequence from conformal invariance these quantities are given by the unitary IR of the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}c(n^3 - n)\delta_{n+m,0} \qquad n, m \in \mathbb{Z} \tag{4.2}$$

with central charge c. An IR is characterised by its highest weight which is a so-called primary field with scaling dimension Δ:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad L_n|\Delta\rangle = 0 \qquad n > 0. \tag{4.3}$$

Excited states, which we will call descendants of $|\Delta\rangle$ with scaling dimension $\Delta + r$, are generated by the operators L_{-r}:

$$L_0(L_{-r_1}L_{-r_2}\ldots)|\Delta\rangle = (\Delta + r)(L_{-r_1}L_{-r_2}\ldots)|\Delta\rangle \qquad 0 < r_1 \leq r_2 \ldots \tag{4.4}$$

where $x = \Delta + r$ is a surface critical exponent giving a contribution to the spectrum (4.1)

$$\mathscr{E}_k = \Delta + r \tag{4.5}$$

with a certain (known) degeneracy $d(\Delta, r)$ and relative parity $(-1)^r$ to $|\Delta\rangle$.

In the case of toroidal boundary conditions the finite-size spectrum is defined by

$$\mathscr{E}_k(P) = \lim_{N\to\infty} \frac{N}{2\pi}(E_k(P, N) - E_F(N)). \tag{4.6}$$

Here $E_F(N)$ is the lowest eigenvalue in sector $H_{0,+}^0$. The critical exponents $x = \mathscr{E}_k(P)$ now are given by the IR $(\Delta + r, \bar{\Delta} + \bar{r})$ of two commuting Virasoro algebras L, $\bar{L}$:

$$x = \Delta + r + \bar{\Delta} + \bar{r} \tag{4.7}$$

with momentum $P = (\Delta + r) - (\bar{\Delta} + \bar{r})$ and spin $s = \Delta - \bar{\Delta}$.

For unitary theories with $c < 1$ the central charge is quantised and there exists only a finite number of highest weight IR to each value of c (Friedan *et al* 1984). Thus, once c is known for a particular model, all the critical exponents determining its singular behaviour or, in the language of the field theory, all the anomalous dimensions of the scaling operators can be computed. As already mentioned, Andrews *et al* (1984) have found a set of exact multicritical points corresponding to this series.

For $c \geq 1$ unitarity does not impose any restrictions on the values of the anomalous dimensions. However, if higher symmetries (e.g. SUSY) occur, one obtains new quantisation conditions, in particular for $c \geq 1$, which allow a classification of such theories.

The $N = 1$ superconformal algebra consists of the Virasoro algebra (4.2) together with the fermionic operators G (Friedan *et al* 1985, Berdshadski *et al* 1985, Eichenherr 1985):

$$\begin{aligned} &[L_m, G_r] = (m/2 - r)G_{m+r} \\ &\{G_r, G_s\} = 2L_{r+s} + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})\delta_{r+s,0} \end{aligned} \tag{4.8}$$

with r, $s \in \mathbb{Z}$ in the Ramond sector (R) and r, $s \in \mathbb{Z} + \frac{1}{2}$ in the Neveu–Schwarz sector

(NS). By the demand for unitarity the central charge is quantised if $c<\frac{3}{2}$:

$$c=\frac{3}{2}-\frac{12}{m(m+2)} \qquad m>2. \tag{4.9}$$

Then the following anomalous dimensions are possible:

$$\Delta_{p,q}=\frac{[p(m+2)-qm]^2-4}{8m(m+2)}+\frac{1-(-1)^{p-q}}{32} \tag{4.10}$$

with $1\leqslant p\leqslant m$, $1\leqslant q\leqslant m+2$; $p-q$ is even in the NS sector and odd in the R sector. For $m=4$ $(c=1)$ we obtain

$$\begin{array}{ll} \text{NS} & 0, \frac{1}{6}, 1, \frac{1}{16} \\ \text{R} & \frac{1}{24}, \frac{3}{8}, \frac{1}{16}, \frac{9}{16} \end{array} \tag{4.11}$$

and for $m=6$ $(c=\frac{5}{4})$ the dimensions are

$$\begin{array}{ll} \text{NS} & 0, \frac{5}{6}, 3, \frac{1}{4}, \frac{1}{12}, \frac{5}{4}, \frac{1}{32}, \frac{33}{32}, \frac{5}{32} \\ \text{R} & \frac{5}{96}, \frac{23}{32}, \frac{3}{32}, \frac{41}{96}, \frac{67}{32}, \frac{5}{16}, \frac{29}{16}, \frac{1}{16}, \frac{9}{16}. \end{array} \tag{4.12}$$

The degeneracies of level r representations $D(\Delta, r)$ are given by the characters of Δ (Goddard *et al* 1986):

$$\begin{aligned} \chi_\Delta^{\mathrm{NS}}(z)&=\Gamma_{p,q}^m(z)\prod_{n=1}^{\infty}\frac{1+z^{n-1/2}}{1-z^n} \\ \chi_\Delta^{\mathrm{R}}(z)&=z^{1/16}\Gamma_{p,q}^m(z)\prod_{n=1}^{\infty}\frac{1+z^n}{1-z^n} \end{aligned} \tag{4.13}$$

where

$$\Gamma_{p,q}^m(z)=\sum_{n\in\mathbb{Z}}\left(z^{[(Mn-l)^2-4]/4M}-z^{[(Mn-\hat{l})^2-4]/4M}\right) \tag{4.14}$$

with

$$\begin{aligned} M&=2m(m+2) \\ l&=(m+2)p-mq \\ \hat{l}&=(m+2)p+mq \end{aligned}$$

(for m, p, q see (4.9) and (4.10)).

$N=1$ supersymmetric models known up to now are the tricritical Ising model $(m=3)$ (Friedan *et al* 1985), the Ashkin-Teller model for special values of the coupling constants $(m=4)$ (Baake *et al* 1987, Yang and Zheng 1987), and the six-states quantum chain (2.1) on the curve S_1S_2 (figure 1) with $m=6$ (Schütz 1987).

The $N=2$ superconformal algebra is achieved by combination of the Virasoro algebra with the U(1) Kac-Moody algebra (Di Vecchia *et al* 1985, Waterson 1986, Boucher *et al* 1986)

$$\left.\begin{aligned} [T_m, T_n]&=\tfrac{1}{3}cm\delta_{m+n,0} \\ [L_m, T_n]&=-nT_{m+n} \end{aligned}\right\} \qquad m, n\in\mathbb{Z} \tag{4.15}$$

together with the following commutation and anticommutation relations:

$$
\begin{aligned}
&[L_m, G_k^\pm] = (m/2 - k) G_{m+k}^\pm \\
&[T_m, G_k^\pm] = \pm G_{m+k}^\pm \\
&\{G_k^+, G_l^-\} = 2L_{k+l} + (k-l) T_{k+l} + \tfrac{1}{3} c (k^2 - \tfrac{1}{4})_{k+l,0}
\end{aligned}
\tag{4.16}
$$

where $k,\ l \in \mathbb{Z}$ for the Ramond sector and $k,\ l \in \mathbb{Z} - \frac{1}{2}$ for the Neveu-Schwarz sector. The IR are given by a highest weight Δ and a charge q related to the eigenvalue of T_0. They are denoted by $(\Delta;\ q)$. Unitary representations exist for

$$
c = 1 + 2(p-1)/(p+2) \qquad p > 0 \tag{4.17}
$$

with IR

$$
\Delta_{l,m}^{(a)} = \frac{l(l+2)}{4(p+2)} + \frac{a^2}{2} - \frac{(m - 2a^2)}{4(p+2)} \qquad q_{l,m}^{(a)} = \frac{m + ap}{p+2}. \tag{4.18}
$$

Here $a = 0$ in the NS sector, $a = \pm\frac{1}{2}$ in the R sector and $0 \leqslant l \leqslant p$. m ranges from $-p$ to p, or according to the symmetry properties of the characters $\chi_{l,m}^{(p,a)}$ (Ravanini and Yang 1987a)

$$
\chi_{l,m}^{(p,a)} = \chi_{p-l,p+2+m}^{(p,a)} = \chi_{l,m+2\mathbb{Z}(p+2)}^{(p,a)} \tag{4.19}
$$

one has $-p-1 \leqslant m \leqslant p+2$. l and m must obey $l - m = 0 \bmod 2$. The characters can be written (Ravanini and Yang 1987a)

$$
\chi_{l,m}^{(p,a)}(\tau, \nu) = \sum_{m'=-p+1}^{p} C_{l,m'}^{(p)}(\tau)\, \theta_{m'(p+2)-mp+2ap,p(p+2)}\left(\frac{\tau}{2}, \frac{\nu}{p+2}\right) \tag{4.20}
$$

where the theta functions are defined by

$$
\begin{aligned}
&\theta_{m,k}(\tau, \nu) = \sum_{n \in Z + m/2k} z^{kn^2} y^{kn} \\
&z = \exp(2\pi i \tau) \qquad y = \exp(2\pi i \nu)
\end{aligned}
\tag{4.21}
$$

and $C_{l,m}^{(p)}$ are the string functions of the affine su(2) algebra which are Hecke indefinite modular forms (see 4.28).

For $c = 1$ (3.19) gives

$$
\begin{aligned}
&\text{NS} \qquad (0;\ 0),\ (\tfrac{1}{6};\ \pm\tfrac{1}{3}) \\
&\text{R}^{\mp} \qquad (\tfrac{3}{8};\ \pm\tfrac{1}{2}),\ (\tfrac{1}{24};\ \pm\tfrac{1}{6}),\ (\tfrac{1}{24};\ \mp\tfrac{1}{6}).
\end{aligned}
\tag{4.22}
$$

Their characters can be decomposed into $N = 1$ characters:

$$
\begin{aligned}
&(0;\ 0)_2^{\text{NS}} = (0)_1^{\text{NS}} + (1)_1^{\text{NS}} \\
&(\tfrac{1}{6};\ q)_2^{\text{NS}} = (\tfrac{1}{6})_1^{\text{NS}} \\
&(\tfrac{3}{8};\ q)_2^{\text{R}} = 2(\tfrac{3}{8})_1^{\text{R}} \\
&(\tfrac{1}{24};\ q)_1^{\text{R}} = (\tfrac{1}{24})_1^{\text{R}}.
\end{aligned}
\tag{4.23}
$$

Notice that

$$
[\tfrac{1}{16}] := (\tfrac{1}{16})_1^{\text{NS}} = (\tfrac{1}{16})_1^{\text{R}} + (\tfrac{9}{16})_1^{\text{R}} \tag{4.24}
$$

is the representation of the twisted $N=2$ superconformal algebra with $c=1$ (Rittenberg and Schwimmer 1987).

$N=2$ supersymmetry is known to appear in the Ashkin–Teller model (Baake *et al* 1987, Yang and Zheng 1987).

Recently Zamolodchikov and Fateev (1985) described a family of two-dimensional conformal field theories, which are symmetric under a fractional spin (non-local) current algebra. They correspond to Z_n-symmetric self-dual models at the critical point and are a generalisation of the critical Ising model ($n=2$), where the Majorana fermion is replaced by non-local analytic fields, called parafermions.

The central charge of the ZF theories is given by

$$c = 2(n-1)/(n+2) \tag{4.25}$$

and the anomalous dimensions characterising these theories are

$$\Delta_{l,m} = \frac{l(l+2)}{4(n+2)} - \frac{m^2}{4n} + \text{integer} + \begin{cases} \frac{1}{2}(l-|m|) & |m|>l \\ 0 & |m|\leqslant l \end{cases}. \tag{4.26}$$

The integers l, m obey $0\leqslant l\leqslant n$, $-n+1\leqslant m\leqslant n$ and $l-m=0 \bmod 2$. For $n=6$ the dimensions are

$$0, \tfrac{3}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{12}, \tfrac{7}{12}, \tfrac{5}{6}, \tfrac{4}{3}, \tfrac{5}{96}, \tfrac{101}{96}, \tfrac{3}{32}, \tfrac{23}{32}, \tfrac{41}{96}. \tag{4.27}$$

The characters $\chi^{\mathrm{ZF}}_{l,m}$ of the representations of the ZF algebra are given by the string functions $c^{(n)}_{l,m}$ (Gepner and Qiu 1987, Gepner 1987)

$$\begin{aligned} \chi_{l,m}(z) &= c^{(n)}_{l,m}(z)\eta(z) \\ &= \eta(z)^{-2} \sum_{\substack{-|x|<y<|x| \\ (x,y)\,\mathrm{or}\,(\frac{1}{2}-x,\frac{1}{2}+y)\in((l+1)/2(n+2),\,m/2n)+\mathbb{Z}^2}} \mathrm{sgn}(x) z^{(k+2)x^2-ky^2} \end{aligned} \tag{4.28}$$

with the Dedekind η function:

$$\eta(z) = z^{1/24} \prod_{n=1}^{\infty} (1-z^n). \tag{4.29}$$

The parafermionic primary fields (order and disorder parameters σ_l, μ_l, $l=0,\ldots,n-1$, which are the generalisations of the spin fields of the Ising model have critical exponents

$$h_l = \frac{l(n-l)}{n(n+2)}. \tag{4.30}$$

All the other primary ZF fields (4.26) (which are infinitely many if $n>4$) are obtained from σ_l, μ_l by applyig successively the generators of the parafermionic current algebra (Zamolodchikov and Fateev 1985, Gepner and Qiu 1987). Due to their Z_n symmetry the fields $(\Delta_{l,m}, \bar{\Delta}_{\bar{l},\bar{m}})$ carry a $Z_n\otimes Z_n$ charge (p, q) which is given by (Gepner and Qiu 1987)

$$(p, q) = \left(\frac{m+\bar{m}}{2}, \frac{m-\bar{m}}{2}\right). \tag{4.31}$$

After this brief review (for more details see Zamolodchikov and Fateev (1985) and Gepner and Qiu (1987)) we specialise to some applications. We notice that $s=\Delta-\bar{\Delta}=Q\tilde{Q}/n=-pq/n$ and conclude $p=Q$, $q=-\tilde{Q} \bmod 6$. Thus, if a representation with charge $(Q, -\tilde{Q})$ appears in the spectrum of $H^{\tilde{Q}}$ (1.1), then the theory predicts it to be found in the sector $H_Q^{\tilde{Q}}$. Invariance of the theory under charge conjugation $p \to -p$ or $Q \to -Q$ respectively and self-duality recover (3.7).

For $n=6$ we obtain the same central charge $c=\frac{5}{4}$ as for $N=1$ supersymmetry with $m=6$. This suggests we cast a glance at the highest weights (4.27) and (4.12) of the corresponding algebras. Using (4.13) and the explicit formula for the string functions $c_{l,m}^{(6)}(q)$ (4.29) we derive the following identities:

$$\begin{aligned}
&[0]^{\mathrm{NS}}+[3]^{\mathrm{NS}}=[0]^{\mathrm{ZF}}+[\tfrac{3}{2}]^{\mathrm{ZF}}\\
&[\tfrac{1}{4}]^{\mathrm{NS}}+[\tfrac{5}{4}]^{\mathrm{NS}}=[\tfrac{1}{4}]^{\mathrm{ZF}}+[\tfrac{3}{4}]^{\mathrm{ZF}}\\
&[\tfrac{5}{6}]^{\mathrm{NS}}=[\tfrac{5}{6}]^{\mathrm{ZF}}+[\tfrac{4}{3}]^{\mathrm{ZF}}\\
&[\tfrac{1}{12}]^{\mathrm{NS}}=[\tfrac{1}{12}]^{\mathrm{ZF}}+[\tfrac{7}{12}]^{\mathrm{ZF}}\\
&[\tfrac{5}{96}]^{\mathrm{R}}=[\tfrac{5}{96}]^{\mathrm{ZF}}+[\tfrac{101}{96}]^{\mathrm{ZF}}\\
&[\tfrac{3}{32}]^{\mathrm{R}}+[\tfrac{67}{32}]^{\mathrm{R}}=[\tfrac{3}{32}]^{\mathrm{ZF}}\\
&[\tfrac{41}{96}]^{\mathrm{R}}=[\tfrac{41}{96}]^{\mathrm{ZF}}\\
&[\tfrac{23}{32}]^{\mathrm{R}}=[\tfrac{23}{32}]^{\mathrm{ZF}}.
\end{aligned} \qquad (4.32)$$

These relations will turn out to be necessary for the interpretation of the spectra in § 5.

5. Operator content of the six-states quantum chain

We are now in a position to determine the operator content of the model (1.1). Applying the Lanczos method (Lanczos 1950) the eigenvalues $E_k(N)$ (4.1) and $E_k(P, N)$ (4.6) can be calculated numerically for chains of length N up to 7 or 8 sites, dependent on the sector. The limits $\mathscr{E}_k$ are then computed using the algorithm of Bulirsch and Stoer (1964). The errors given are very subjective due to the nature of the extrapolation algorithms for this kind of problem. They are obtained by studying the variation of the approximants with a free parameter entering the algorithm of Bulirsch and Stoer (for more details see Henkel and Schütz (1988)). The calculations have been done in all the sectors for up to 30 levels per sector. In order not to flood the paper with tables not all the numerical results will be presented. However, it turns out that the general agreement with (4.11), (4.12) and (4.27) is excellent. A typical example is the spectrum for periodic boundary conditions given in table 5. These results can be discussed in terms of representations of the $N=1$ and $N=2$ superconformal and the ZF algebra. Applying the above-mentioned methods we determine the critical properties of H in three steps.

(i) The normalisation factor ξ is calculated following the method of von Gehlen *et al* (1986), from the lowest eigenvalue with $P=2$ in sector $H_{0,+}^{0}$; see tables 2 and 3.

(ii) The central charge c can be determined from the correction to the ground-state energy (Blöte *et al* 1986, Affleck 1986). The free energy per site at the critical point for periodic boundary condition is

$$-E_{\mathrm{F}}(N)/N = A + \tfrac{1}{6}\pi c N^{-2} + \ldots. \qquad (5.1)$$

This relation provides a direct method of determining c; see tables 2 and 3 and figure 1. At the ZF point we find $c=1.25(1)$ and at the vector Potts point $c=1.00(1)$.

Table 2. Normalisation ξ and central charge c along the line $\delta=1$ for various values of ε.

ε	-0.95	$-\frac{2}{3}$	$-\frac{1}{3}$	-0.1	0.1	0.2	1
ξ	0.126 (3)	0.82 (1)	1.59 (1)	2.11 (1)	2.51 (1)	2.73 (1)	4.132
c	1.00 (2)	1.00 (3)	1.11 (4)	1.21 (3)	1.24 (2)	1.24 (2)	1.20 (2)

Table 3. Normalisation ξ and central charge c in the ZF point ($\varepsilon=1/\sqrt{3}$, $\delta=\frac{1}{2}$) and vector Potts point ($\varepsilon=\delta=0$).

ε, δ	$1/\sqrt{3}, \frac{1}{2}$	0, 0
ξ	3.002 (2)	1.474 (2)
c	1.25 (1)	1.00 (1)

(iii) We compare the possible anomalous dimensions (4.11), (4.12) and (4.27) with the numerical extrapolants $\mathscr{E}_k$ (Schütz 1987, von Gehlen *et al* 1988, table 5(a–f)). Additionally it is necessary to control the degeneracies of the excited states. With that it is ascertained which algebra describes the underlying symmetry and which of the possible representations are actually realised.

In order to study the appearance of representations of these algebras, we first examine the operator content in the region B with $c=1$. It has been conjectured and checked numerically (von Gehlen *et al* 1988) that the operator content remains constant at $\lambda=1$ in the $c=1$ region and we present it in table 4(a–d), column 2 for $\lambda=1$ in terms of representations of the Virasoro algebra.

We start with the discussion of the spectrum for free boundaries. The spectrum H^F is given by the IR of one Virasoro algebra. Comparison of the anomalous dimensions (table 4(a), column 2) with those possible for $N=1$ supersymmetry (4.11) and with the degeneracies calculated from the character formulae (4.13) shows that the spectrum is $N=1$ supersymmetric if different sectors are combined (column 3). The combination of sectors, if necessary even for different (toroidal) boundary conditions, will also prove useful for the interpretation of the spectra below. We find that the spectrum of H^F in the $c=1$ region B is composed by the operators $(0)_1^{\mathrm{NS}}$, $(1)_1^{\mathrm{NS}}$ and $2(\frac{1}{16})_1^{\mathrm{NS}}$. They form a closed subalgebra.

We now investigate the occurrence of higher symmetries. Making use of the decomposition (4.23) one finds $N=2$ SUSY (column 4) by further combination of sectors. The observation of $N=2$ supersymmetry is due to the simultaneous appearance of the $N=1$ superconformal algebra and the U(1) Kac–Moody algebra known to be present in the $c=1$ region of any n-states quantum chain (von Gehlen *et al* 1988).

The question arises whether the combination of all sectors is given by the vacuum representation $[0]^S$ of higher algebra (column 5). We keep this idea in our mind and turn to the consideration of the toroidal spectrum.

Here the representations of two commuting Virasoro algebras determine the possible critical exponents. Studying the sectors of the periodic spectrum (table 4(b), column 2) the highest weights (4.11) are recovered, but it is not supersymmetric, as the degeneracies do not agree with those from the character formulae (4.13). However, the combination of certain spectra for periodic and antiperiodic boundary conditions $H^{\mathrm{SUSY}}=H^0+H^3$ shows the 'correct' degeneracies and thus is $N=1$ supersymmetric (column 3). H^{SUSY} is given by

$$(0,0)_1^{\mathrm{NS}} \oplus (1,0)_1^{\mathrm{NS}} \oplus (0,1)_1^{\mathrm{NS}} \oplus (1,1)_1^{\mathrm{NS}} \oplus 2(\tfrac{1}{6},\tfrac{1}{6})_1^{\mathrm{NS}} \oplus 4(\tfrac{3}{8},\tfrac{3}{8})_1^{\mathrm{R}} \oplus 2(\tfrac{1}{24},\tfrac{1}{24})_1^{\mathrm{R}}. \tag{5.2}$$

As above, further combination of sectors yields $N=2$ SUSY (table 4(b), column 4).

Table 4. (a) Operator content in region B for free boundary conditions in the sector H_U^B in terms of representations of the Virasoro algebra $(\Delta)^{\mathrm{V}}$ (von Gehlen *et al* 1988), the $N=1$ superconformal algebra $(\Delta)^{\mathrm{NS,R}}$, the $N=2$ superconformal algebra $(\Delta)_2^{\mathrm{NS,R,T}}$ and the multiplet fields (5.5) $[\Delta]^{\mathrm{S}}$. (b) As (a) for H^{SUSY} (periodic and antiperiodic boundary conditions). (c) As (a) for H^C (cyclic boundary conditions). (d) As (a) for H^T (twisted boundary conditions).

(a) H_U^F	$(\Delta)^{\mathrm{V}}$	$(\Delta)_1^{\mathrm{NS}}$	$(\Delta)_2^{\mathrm{NS}}$	$[\Delta]^S$
$H_{0,+}^F$	$\bigoplus_{k\geq 0}(4k^2)^{\mathrm{V}} \oplus \bigoplus_{r\geq 1}(6r^2)^{\mathrm{V}} =: \{0\}$	$(0)_1^{\mathrm{NS}}$	$(0)_2^{\mathrm{NS}}$	$[0]^S$
$H_{3,+}^F$	$\bigoplus_{r\geq 0}(\tfrac{3}{2}(2r+1)^2)^{\mathrm{V}} =: \{\tfrac{3}{2}\}$			
$H_{0,-}^F$	$\bigoplus_{k\geq 0}((2k+1)^2)^{\mathrm{V}} \oplus \bigoplus_{r\geq 1}(6r^2)^{\mathrm{V}} =: \{1\}$	$(1)_1^{\mathrm{NS}}$		
$H_{3,-}^F$	$= H_{3,+}^F$			
$H_1^F \oplus H_5^F$	$2[\bigoplus_{r\in\mathbb{Z}}(6(r+\tfrac{1}{6})^2)^{\mathrm{V}}] =: 2\{\tfrac{1}{6}\}$	$2(\tfrac{1}{6})_1^{\mathrm{NS}}$	$2(\tfrac{1}{6})_2^{\mathrm{NS}}$	
$H_2^F \oplus H_4^F$	$2[\bigoplus_{r\in\mathbb{Z}}(6(r+\tfrac{1}{3})^2)^{\mathrm{V}}] =: 2\{\tfrac{2}{3}\}$			

(b) H_U^B	$(\Delta,\bar\Delta)^{\mathrm{V}}$	$(\Delta,\bar\Delta)_1^{\mathrm{NS,R}}$	$(\Delta,\bar\Delta)_2^{\mathrm{NS,R}}$
$H_{0,+}^0$	$\{0,0\}\oplus\{1,1\}\oplus 2\{\tfrac{3}{2},\tfrac{3}{2}\}$	$(0,0)_1^{\mathrm{NS}} \oplus (1,1)_1^{\mathrm{NS}}$	$(0,0)_2^{\mathrm{NS}}$
$H_{3,+}^3$	$\{0,\tfrac{3}{2}\}\oplus\{\tfrac{3}{2},0\}\oplus\{1,\tfrac{3}{2}\}+\{\tfrac{3}{2},1\}$		
$H_{0,-}^0$	$\{0,1\}\oplus\{1,0\}\oplus 2\{\tfrac{3}{2},\tfrac{3}{2}\}$	$(0,1)_1^{\mathrm{NS}} \oplus (1,0)_1^{\mathrm{NS}}$	
$H_{3,-}^3$	$= H_{3,+}^3$		
$H_2^0 \oplus H_4^0$	$2[\{\tfrac{1}{6},\tfrac{1}{6}\}\oplus\{\tfrac{2}{3},\tfrac{2}{3}\}]$	$2(\tfrac{1}{6},\tfrac{1}{6})_1^{\mathrm{NS}}$	$2(\tfrac{1}{6},\tfrac{1}{6})_2^{\mathrm{NS}}$
$H_1^3 \oplus H_5^3$	$2[\{\tfrac{1}{6},\tfrac{2}{3}\}\oplus\{\tfrac{2}{3},\tfrac{1}{6}\}]$		
$H_{3,+}^0 \oplus H_{0,+}^3$	$2[\bigoplus_{r\geq 0}(\tfrac{3}{2}(r+\tfrac{1}{2})^2, \tfrac{3}{2}(r+\tfrac{1}{2})^2)^V] =: 2\{\tfrac{3}{8},\tfrac{3}{8}\}$	$2(\tfrac{3}{8},\tfrac{3}{8})_1^{\mathrm{R}}$	$(\tfrac{3}{8},\tfrac{3}{8})_2^{\mathrm{R}}$
$H_{3,-}^0 \oplus H_{0,-}^3$	$= H_{3,+}^0 \oplus H_{0,+}^3$	$2(\tfrac{3}{8},\tfrac{3}{8})_1^{\mathrm{R}}$	
$H_1^0 \oplus H_5^0$	$2[\bigoplus_{r\geq 0}(\tfrac{3}{2}(r+\tfrac{1}{6})^2, \tfrac{3}{2}(r+\tfrac{1}{6})^2)^V \oplus \bigoplus_{r\geq 0}(\tfrac{3}{2}(r+\tfrac{5}{6})^2, \tfrac{3}{2}(r+\tfrac{5}{6})^2)^V] =: 2[\{\tfrac{1}{24},\tfrac{1}{24}\}+\{\tfrac{25}{24},\tfrac{25}{24}\}]$	$2(\tfrac{1}{24},\tfrac{1}{24})_1^{\mathrm{R}}$	$2(\tfrac{1}{24},\tfrac{1}{24})_2^{\mathrm{R}}$
$H_2^3 \oplus H_4^3$	$2[\{\tfrac{1}{24},\tfrac{25}{24}\}+\{\tfrac{25}{24},\tfrac{1}{24}\}]$		

(c) H_U^B	$(\Delta,\bar\Delta)_2^{\mathrm{NS,R}}$	$[\Delta,\bar\Delta]^S$
$H_{0,+}^0 \oplus H_{0,-}^0 \oplus H_{3,+}^3 \oplus H_{3,-}^3$	$(0,0)_2^{\mathrm{NS}}$	$[0,0]^S$
$H_2^0 \oplus H_4^0 \oplus H_1^3 \oplus H_5^3$	$4(\tfrac{1}{6},\tfrac{1}{6})_2^{\mathrm{NS}}$	
$H_1^1 \oplus H_5^1 \oplus H_1^5 \oplus H_5^5$	$2[(0,\tfrac{1}{6})_2^{\mathrm{NS}} \oplus (\tfrac{1}{6},0)_2^{\mathrm{NS}}]$	
$H_2^2 \oplus H_4^2 \oplus H_2^4 \oplus H_4^4$		
$H_{3,+}^0 \oplus H_{3,-}^0 \oplus H_{0,+}^3 \oplus H_{0,-}^3$	$(\tfrac{3}{8},\tfrac{3}{8})_2^{\mathrm{R}}$	$[\tfrac{1}{24},\tfrac{1}{24}]^S$
$H_1^0 \oplus H_5^0 \oplus H_2^3 \oplus H_4^3$	$2(\tfrac{1}{24},\tfrac{1}{24})_2^{\mathrm{R}}$	
$H_2^1 \oplus H_4^1 \oplus H_1^4 \oplus H_1^2$	$2[(\tfrac{1}{24},\tfrac{3}{8})_2^{\mathrm{R}} \oplus (\tfrac{3}{8},\tfrac{1}{24})]_2^{\mathrm{R}}$	
$H_5^2 \oplus H_2^5 \oplus H_4^5 \oplus H_5^4$		

Table 4. (continued)

$(d)\ H^B_U\,(\Delta,\bar\Delta)^{\rm V}$			$[\Delta,\bar\Delta]^T_2$
H^{OC}_{++}	$\bigoplus_{k\in\mathbb{Z}}((8k+1)^2/16,\,(8k+1)^2/16)^{\rm V}\oplus\bigoplus_{k\in\mathbb{Z}}((8k+3)^2/16,\,(8k+3)^2/16)^{\rm V}=:\{\frac{1}{16},\frac{1}{16}\}\oplus\{\frac{9}{16},\frac{9}{16}\}$		$[\frac{1}{16},\frac{1}{16}]^T_2$
H^{OC}_{+-}	$\{\frac{1}{16},\frac{9}{16}\}\oplus\{\frac{9}{16},\frac{1}{16}\}$		
H^{3C}_{-+}		$=H^{0C}_{+-}$	$[\frac{1}{16},\frac{1}{16}]^T_2$
H^{3C}_{--}		$=H^{0C}_{++}$	
H^{0C}_{-+}	$=H^{3C}_{+-}$	$=H^{0C}_{+-}$	$[\frac{1}{16},\frac{1}{16}]^T_2$
H^{0C}_{--}	$=H^{3}_{++}$	$=H^{0C}_{++}$	

Fields with fractional spin $s=m/6$ appear in the spectra for the remaining cyclic boundary conditions (table 4(c)). In the spectrum

$$H^c=\sum_{\tilde Q=0}^{5}H^{\tilde Q} \tag{5.3}$$

we find the operator content (in terms of $N=1$ SUSY IR):

$$\begin{aligned}(0,0)^{\rm NS}_1&\oplus(1,0)^{\rm NS}_1\oplus(0,1)^{\rm NS}_1\oplus(1,1)^{\rm NS}_1\oplus4(\tfrac16,\tfrac16)^{\rm NS}_1\\&\oplus2\{(\tfrac16,0)^{\rm NS}_1\oplus(0,\tfrac16)^{\rm NS}_1\oplus(\tfrac16,1)^{\rm NS}_1\oplus(1,\tfrac16)^{\rm NS}_1\}\\&\oplus4\{(\tfrac1{24},\tfrac1{24})^{\rm R}_1\oplus(\tfrac38,\tfrac1{24})^{\rm R}_1\oplus(\tfrac1{24},\tfrac38)^{\rm R}_1\oplus(\tfrac38,\tfrac38)^{\rm R}_1\}.\end{aligned} \tag{5.4}$$

Now we come to the problem of higher symmetries in the model. We introduce multiplet fields $[h^a_l]$ constructed out of (4.19):

$$\chi^a_l=\sum_{m=-p-1}^{p+2}\chi^a_{l,m}=\chi^a_{p-l} \tag{5.5}$$

and we get for $c=1$ $(p=1)$

$$\begin{aligned}a=0\qquad&[0]^S=(0;0)^{\rm NS}_2\oplus(\tfrac16;\tfrac13)^{\rm NS}_2\oplus(\tfrac16;-\tfrac13)^{\rm NS}_2\\&\qquad=(0)^{\rm NS}_1\oplus2(\tfrac16)^{\rm NS}_1\oplus(1)^{\rm NS}_1\\a=-\tfrac12\qquad&[\tfrac1{24}]^S=(\tfrac1{24};\tfrac16)^{\rm R}_2\oplus(\tfrac1{24};-\tfrac16)^{\rm R}_2\oplus(\tfrac38;\tfrac12)^{\rm R}_2\\&\qquad=2(\tfrac1{24})^{\rm R}_1\oplus2(\tfrac38)^{\rm R}_1.\end{aligned} \tag{5.6}$$

Investigating table 4 the operator content of H^F and H^c turns out to be given in terms of these multiplets. We find

$$H^F\qquad[0]^S \tag{5.7}$$

and

$$H^c\qquad[0,0]^S\oplus[\tfrac1{24},\tfrac1{24}]^S.$$

Thus H^F and H^c are part of a higher, yet unknown, algebra which includes the $N=2$ superconformal algebra as a subalgebra. Its representations are given by the character expressions (5.5).

The representations which are still missing according to table 2 are found in the spectra H^T with twisted boundary conditions (table 4(d)). Combination of the sectors leads to the IR (4.24) of the twisted $N=2$ superconformal algebra.

We now turn to the ZF point $\varepsilon = 1/\sqrt{3}$, $\delta = \frac{1}{2}$ on the supersymmetric line S_1S_2 with $c = \frac{5}{4}$. While central charge and anomalous dimensions remain constant along the curve S_1S_2, they change moving away from it.

Again we start the discussion by examining the spectrum for free boundary conditions. The same combination of sectors which led to $N = 1$ SUSY in the $c = 1$ region (table 4(a)) shows that (Schütz 1987)

$$(0)_1^{NS} \oplus 2(\tfrac{5}{6})_1^{NS} \oplus (3)_1^{NS} \tag{5.8}$$

gives the operator content for free boundary conditions. As above, these operators form a closed subalgebra. Combining the sectors with positive and negative charge conjugation one recovers ZF symmetry:

$$\begin{aligned} &\mathscr{E}_{0,+} \oplus \mathscr{E}_{0,-} = (0)^{ZF} \\ &\mathscr{E}_1 = \mathscr{E}_5 = (\tfrac{5}{6})^{ZF} \\ &\mathscr{E}_2 = \mathscr{E}_4 = (\tfrac{4}{3})^{ZF} \\ &\mathscr{E}_{3,+} \oplus \mathscr{E}_{3,-} = (\tfrac{3}{2})^{ZF}. \end{aligned} \tag{5.9}$$

The charge Q of $(\Delta_{l,m}^{ZF})$ (4.26) is given by $m/2 \bmod 6$.

In the case of toroidal boundary conditions we observe a structure similar to that in the $c = 1$ region. First we turn to the representations of $N = 1$ SUSY. H^0 (see table 5(a–f)) itself is not $N = 1$ supersymmetric. Notice the 'wrong' degeneracies D for the descendants of the magnetic operator $(\frac{5}{96}, \frac{5}{96})$ (table 5(d)) and an additional 'forbidden' field, $(\frac{101}{96}, \frac{101}{96})$. Once again, the combination with the antiperiodic spectrum clears the situation. $H^{SUSY} = H^0 + H^3$ (table 5(d)) gives the desired degeneracies of the excited states of $(\frac{5}{96}, \frac{5}{96})$. In terms of $N = 1$ IR the operator content of the spectra H^{SUSY}, H^c and $H^T = \Sigma_{q=0}^5 H^{qc}$ is given by

$$\begin{aligned} H^{SUSY} \quad & (0,0) \oplus (3,0) \oplus (0,3) \oplus (3,3) \oplus 2(\tfrac{5}{6},\tfrac{5}{6}) \oplus (\tfrac{1}{4},\tfrac{1}{4}) \\ & \oplus (\tfrac{5}{4},\tfrac{1}{4}) \oplus (\tfrac{1}{4},\tfrac{5}{4}) \oplus (\tfrac{5}{4},\tfrac{5}{4}) \oplus 2(\tfrac{1}{12},\tfrac{1}{12}) \\ & \oplus 2\{(\tfrac{5}{96},\tfrac{5}{96}) \oplus 2(\tfrac{23}{32},\tfrac{23}{32}) \oplus (\tfrac{3}{32},\tfrac{3}{32}) \oplus (\tfrac{67}{32},\tfrac{3}{32}) \\ & \oplus (\tfrac{3}{32},\tfrac{67}{32}) \oplus (\tfrac{67}{32},\tfrac{67}{32}) \oplus 2(\tfrac{41}{96},\tfrac{41}{96})\} \end{aligned} \tag{5.10}$$

$$\begin{aligned} H^c \quad & (0,0) \oplus (3,0) \oplus (0,3) \oplus (3,3) \oplus 4(\tfrac{5}{6},\tfrac{5}{6}) \\ & \oplus 2\{(\tfrac{5}{6},0) \oplus (0,\tfrac{5}{6}) \oplus (\tfrac{5}{6},3) \oplus (3,\tfrac{5}{6})\} \\ & \oplus (\tfrac{1}{4},\tfrac{1}{4}) \oplus (\tfrac{5}{4},\tfrac{1}{4}) \oplus (\tfrac{1}{4},\tfrac{5}{4}) \oplus (\tfrac{5}{4},\tfrac{5}{4}) \oplus 4(\tfrac{1}{12},\tfrac{1}{12}) \\ & \oplus 2\{(\tfrac{1}{12},\tfrac{1}{4}) \oplus (\tfrac{1}{4},\tfrac{1}{12}) \oplus (\tfrac{1}{12},\tfrac{5}{4}) \oplus (\tfrac{5}{4},\tfrac{1}{12})\} \\ & \oplus 4\{(\tfrac{5}{96},\tfrac{5}{96}) \oplus (\tfrac{23}{32},\tfrac{23}{32}) \oplus (\tfrac{5}{96},\tfrac{23}{32}) \oplus (\tfrac{23}{32},\tfrac{5}{96})\} \\ & \oplus 2\{(\tfrac{3}{32},\tfrac{3}{32}) \oplus (\tfrac{67}{32},\tfrac{3}{32}) \oplus (\tfrac{3}{32},\tfrac{67}{32}) \oplus (\tfrac{67}{32},\tfrac{67}{32}) \\ & \oplus 4(\tfrac{41}{96},\tfrac{41}{96}) \oplus 2\{(\tfrac{41}{96},\tfrac{3}{32}) \oplus (\tfrac{3}{32},\tfrac{41}{96}) \oplus (\tfrac{41}{96},\tfrac{67}{32}) \oplus (\tfrac{67}{32},\tfrac{41}{96})\}\} \end{aligned} \tag{5.11}$$

$$\begin{aligned} H^T \quad & 3\{(\tfrac{1}{32},\tfrac{1}{32}) \oplus (\tfrac{33}{32},\tfrac{1}{32}) \oplus (\tfrac{1}{32},\tfrac{33}{32}) \oplus (\tfrac{33}{32},\tfrac{33}{32})\} \oplus 6(\tfrac{5}{32},\tfrac{5}{32}) \\ & \oplus 6\{(\tfrac{1}{16},\tfrac{1}{16}) \oplus (\tfrac{9}{16},\tfrac{1}{16}) \oplus (\tfrac{1}{16},\tfrac{9}{16}) \oplus (\tfrac{9}{16},\tfrac{9}{16})\} \\ & \oplus 6\{(\tfrac{5}{16},\tfrac{5}{16}) \oplus (\tfrac{29}{16},\tfrac{5}{16}) \oplus (\tfrac{5}{16},\tfrac{29}{16}) \oplus (\tfrac{29}{16},\tfrac{29}{16})\}. \end{aligned} \tag{5.12}$$

Table 5. (a–f) Operator content along line S_1S_2 in terms of IR $(\Delta + r, \bar{\Delta} + \bar{r})$ of $N = 1$ SUSY for toroidal boundary conditions. P denotes the momentum of a state, x the critical exponent with degeneracy $D(\Delta, r;\ \bar{\Delta}, \bar{r})$. $\mathscr{E}^B_U$ are the finite-size scaling estimates in the sector H^B_U at the ZF point Z ($\varepsilon = 1/\sqrt{3}$, $\delta = \frac{1}{2}$). Exponents marked by$^+$ are fixed by definition, † denotes the thermal exponent and * the magnetic exponents (order parameters) x_Q in the sector Q.

(a)

P	x	$(0,0)$	$(\frac{3}{2},\frac{3}{2})$	$(\frac{1}{4},\frac{1}{4})$	$(\frac{3}{4},\frac{3}{4})$	$(\frac{5}{4},\frac{5}{4})$	$(\frac{7}{4},\frac{7}{4})$	$(3,3)$	$(\frac{7}{2},\frac{7}{2})$	$\mathscr{E}^0_{0,+}(P)$
0	0.0	1	—	—	—	—	—	—	—	0.0$^+$
	0.5	—	—	1	—	—	—	—	—	0.501 (2)†
	1.5	—	—	—	1	—	—	—	—	1.50 (5)
	2.5	—	—	1	—	1	—	—	—	2.515 (3), 2.6 (1)
	3.0	—	1	—	—	—	—	—	—	3.00 (2)
	3.5	—	—	—	1	—	1	—	—	3.6 (1), 3.6 (1)
	4.0	1	—	—	—	—	—	—	—	4.0 (1)
	4.5	—	—	4	—	1	—	—	—	4.40 (2), 4.6 (3), 4.6 (4)
1	1.5	—	—	1	—	—	—	—	—	1.50 (1)
	2.5	—	—	—	1	—	—	—	—	2.64 (1)
	3.5	—	—	2	—	1	—	—	—	3.4 (1), 3.50 (5), 3.7 (1)
2	2.0	1	—	—	—	—	—	—	—	2.0$^+$
	2.5	—	—	2	—	—	—	—	—	2.44 (4), 2.63 (1)
3	3.0	1	—	—	—	—	—	—	—	3.00 (1)
	3.5	—	—	3	—	—	—	—	—	3.44 (4)

(b)

P	x	$(0,3)$	$(3,0)$	$(\frac{3}{2},\frac{7}{2})$	$(\frac{7}{2},\frac{3}{2})$	$(\frac{1}{4},\frac{5}{4})$	$(\frac{5}{4},\frac{1}{4})$	$(\frac{3}{4},\frac{7}{4})$	$(\frac{7}{4},\frac{3}{4})$	$\mathscr{E}^0_{0,-}(P)$
0	2.5	—	—	—	—	1	1	—	—	2.505 (5), 2.505 (5)
	3.5	—	—	—	—	—	—	1	1	3.48 (1), 3.48 (1)
	4.5	—	—	—	—	2	2	—	—	4.2 (1), 4.2 (1), 4.5 (1), 4.5 (1)
1	1.5	—	—	—	—	—	1	—	—	1.507 (1)
	2.5	—	—	—	—	—	—	—	1	2.47 (1)
	3.5	—	—	—	—	2	1	—	—	3.47 (1), 3.55 (1), 3.55 (1)
2	2.5	—	—	—	—	—	1	—	—	2.50 (5)
	3.5	—	—	—	—	—	—	—	2	3.45 (1), 3.6 (1)
	4.0	—	—	—	1	—	—	—	—	4.1 (2)
	4.5	—	—	—	—	3	3	—	—	4.50 (5), 4.5 (2)
3	3.0	—	1	—	—	—	—	—	—	3.004 (5)
	3.5	—	—	—	—	—	3	—	—	3.4 (1), 3.50 (4), 3.5 (1)
	4.5	—	—	—	—	—	—	—	3	4.4 (1), 4.4 (1)

(c)

P	x	$(\frac{1}{12},\frac{1}{12})$	$(\frac{7}{12},\frac{7}{12})$	$(\frac{5}{6},\frac{5}{6})$	$(\frac{4}{3},\frac{4}{3})$	$\mathscr{E}^0_2(P)$
0	0.1667	1	—	—	—	0.1666 (1)*
	1.1667	—	1	—	—	1.172 (2)
	1.6667	—	—	1	—	1.666 (2)
	2.1667	1	—	—	—	2.164 (2)
	2.6667	—	—	—	1	2.70 (2)
	3.1667	—	4	—	—	3.13 (2), 3.16 (2), 3.24 (4), 3.26 (4)
	3.6667	—	—	1	—	3.64 (4)
	4.1667	9	—	—	—	4.0 (2), 4.1 (1), 4.12 (5), 4.16 (5), 4.2 (3)
	4.6667	—	—	—	1	4.78 (5)
1	1.1667	1	—	—	—	1.165 (2)
	2.1667	—	2	—	—	2.14 (3), 2.25 (3)
	2.6667	—	—	1	—	2.65 (3)

Table 5. (continued)

(*d*)

P	x	$(\frac{5}{96}, \frac{5}{96})$	$(\frac{41}{96}, \frac{41}{96})$	$\mathcal{E}_1^0(P)$	$\mathcal{E}_2^3(P)$
0	0.1042	1	—	0.103 (1)*	
	0.8542	—	1	0.857 (3)	0.853 (4)
	2.1042	4	—	2.10 (1), 2.17 (1)	2.10 (5), 2.10 (5)
	2.8542	—	4	2.857 (3), 2.857 (3), 2.88 (2), 3.00 (5) 3.9 (2), 4.0 (2), 4.06 (3) 4.06 (3), 4.07 (4), 4.16 (4)	2.77 (3), 2.82 (3), 2.82 (3), 2.84 (2), 4.0 (1), 4.0(1), 4.0 (1), 4.0 (1)
1	1.1042	2	—	1.109 (3)	1.105 (5)
	1.8542	—	2	1.860 (5), 2.0 (1)	1.80 (2), 1.84 (1)
	3.1042	8	—	2.9 (3), 3.08 (3), 3.20 (5), 3.2 (1)	3.0 (1), 3.04 (4), 3.12 (2), 3.20 (5)
2	2.1042	4	—	2.10 (2), 2.12 (2)	2.09 (1), 2.13 (1)
	2.8542	—	4	2.80 (3), 2.86 (3), 3.0 (1), 3.0 (1)	2.75 (5), 2.78 (3), 2.85 (1) 2.9 (1)

(*e*)

P	x	$(\frac{3}{32}, \frac{3}{32})$	$(\frac{67}{32}, \frac{67}{32})$	$(\frac{23}{32}, \frac{23}{32})$	$\mathcal{E}^0_{3,+}(P)$
0	0.1875	1	—	—	0.1873 (1)*
	1.4375	—	—	1	1.441 (5)
	2.1875	1	—	—	2.185 (1)
	3.4375	—	—	4	3.40 (2), 3.40 (2), 3.47 (1), 3.48 (1)
	4.1875	4	1	—	4.14 (5), 4.14 (5), 4.18 (2), 4.18 (2), 4.3 (1)
	5.4375	—	—	9	5.2 (1), 5.2 (1), 5.2 (2), 5.2 (2), 5.2 (2), 5.40 (5), 5.40 (5), 5.5 (1), 5.5 (1), 5.6 (3)
1	1.1875	1	—	—	1.188 (1)
	2.4375	—	—	2	2.40 (2), 2.46 (2)
	3.1875	2	—	—	3.18 (1), 3.18 (1)
	4.4375	—	—	6	4.3 (1), 4.36 (5), 4.37 (3), 4.45 (5)

(*f*)

P	x	$(\frac{23}{32}, \frac{23}{32})$	$(\frac{3}{32}, \frac{67}{32})$	$(\frac{67}{32}, \frac{3}{32})$	$\mathcal{E}^0_{3,-}(P)$
0	1.4375	1	—	—	1.44 (1)
	3.4375	4	—	—	3.40 (2), 3.40 (2), 3.47 (1), 3.48 (1)
	4.1875	—	2	2	4.14 (5), 4.18 (2), 4.18 (2), 4.32 (5)
	5.4375	9	—	—	5.2 (1), 5.2 (1), 5.2 (2), 5.2 (2), 5.2 (2), 5.40 (5), 5.40 (5), 5.5 (1), 5.5 (1), 5.6 (3)
1	2.4375	2	—	—	2.40 (2), 2.46 (2)
	3.1875	—	—	1	3.18 (1)
	4.4375	6	—	—	4.3 (1), 4.36 (5), 4.37 (3), 4.45 (5)
2	2.1875	—	—	1	2.18 (1)
	3.4375	3	—	—	3.3 (1), 3.45 (1), 3.46 (2)
	4.1875	—	—	2	4.17 (3), 4.2 (1)

Again all possible representations of the $N=1$ superconformal algebra with spin $s=m/6$ are realised by suitable combinations of sectors.

Now we settle the connection with ZF invariance. Equation (4.27) explains the appearance of the additional field $(\frac{101}{96}, \frac{101}{96})$ in the periodic spectrum as a representation of the ZF algebra and using (4.32) we find that H^c is built from ZF IR. In fact, evaluating the character formula (4.28) and comparing with the degeneracies obtained numerically shows that all sectors with cyclic boundary conditions have ZF symmetry with $Z_6 \otimes Z_6$ charge $(Q, -\tilde{Q})$ as predicted (4.31). Charge conjugation decomposes the l even doublet and l odd singlet (4.32) with charge 0 and 3, respectively, into $N=1$ supersymmetric representations. The representations in the twisted sector can be constructed out of representation of a twisted ZF symmetry (Zamolodchikov and Fateev 1986, Ravanini and Yang 1987b).

The multiplet fields $\{h_l\}$ with highest weight (4.30) of the form

$$\chi_l = \sum_{m=-n+1}^{n} \chi_{l,m}^{\mathrm{ZF}} = \chi_{n-l} \tag{5.13}$$

where $0 \leqslant l \leqslant n$, $-n-1 \leqslant m \leqslant n$ and $l-m=0 \bmod 2$ generate the ZF Hilbert space and describe the spectrum H^c. We have

$$\begin{aligned}
l=0 \qquad & \{0\} = (0)^{\mathrm{ZF}} \oplus 2(\tfrac{5}{6})^{\mathrm{ZF}} \oplus 2(\tfrac{4}{3})^{\mathrm{ZF}} \oplus (\tfrac{3}{2})^{\mathrm{ZF}} \\
& \quad = (0)_1^{\mathrm{NS}} \oplus 2(\tfrac{5}{6})_1^{\mathrm{NS}} \oplus (3)_1^{\mathrm{NS}} \\
l=1 \qquad & \{\tfrac{5}{96}\} = 2((\tfrac{5}{96})^{\mathrm{ZF}} \oplus (\tfrac{23}{32})^{\mathrm{ZF}} \oplus (\tfrac{101}{96})^{\mathrm{ZF}}) \\
& \quad = 2((\tfrac{5}{96})_1^{R} \oplus (\tfrac{23}{32})_1^{R}) \\
l=2 \qquad & \{\tfrac{1}{12}\} = (\tfrac{1}{4})^{\mathrm{ZF}} \oplus 2(\tfrac{1}{12})^{\mathrm{ZF}} \oplus 2(\tfrac{7}{12})^{\mathrm{ZF}} \oplus (\tfrac{3}{4})^{\mathrm{ZF}} \\
& \quad = (\tfrac{1}{4})_1^{\mathrm{NS}} \oplus 2(\tfrac{1}{12})_1^{\mathrm{NS}} \oplus (\tfrac{5}{4})_1^{\mathrm{NS}} \\
l=3 \qquad & \{\tfrac{3}{32}\} = 2((\tfrac{3}{32})^{\mathrm{ZF}} \oplus 2(\tfrac{41}{96})^{\mathrm{ZF}}) \\
& \quad = 2((\tfrac{3}{32})_1^{\mathrm{R}} \oplus 2(\tfrac{41}{96})_1^{\mathrm{R}} \oplus (\tfrac{67}{32})_1^{\mathrm{R}}).
\end{aligned} \tag{5.14}$$

The spectrum is built by the order and corresponding disorder fields (4.30) σ_l, μ_l, $l=0, \ldots, n-1$, and the independent fields (4.26) obtained by applying the generators of the parafermionic algebra.

As above, the combination of sectors, i.e. interpretation of primary fields of one algebra as descendants of highest weight IR of a higher algebra, leads to higher symmetries. Due to the constancy of the critical exponents $N=1$ superconformal invariance and ZF symmetry do not vanish even if one moves away from the ZF point along curve S_1S_2.

6. Summary and conclusions

Applying finite-size scaling methods we numerically examined the operator content of the six-states quantum chain given by (1.1) at the critical temperature $\lambda = 1$ in the domain of ferromagnetic interaction. We gave a phenomenological survey over the critical behaviour of the model. In the context of the classification of two-dimensional conformally invariant field theories we discussed the infinite Lie algebras which describe the system.

Numerical diagonalisation of the Hamiltonian $H(\varepsilon, \delta)$ (1.1) for different values of the coupling constants and application of finite-size scaling methods show that H defines a model with varying central charge $c \geqslant 1$ of the Virasoro algebra. We found a region B (figure 1) in the space of coupling constants where $c=1$. The critical exponents in this region remain constant with respect to the couplings and are given by the Gauss model. In a region A with $c=\frac{5}{4}$ runs a curve S_1S_2 with constant exponents. Here we present the full operator content for free and all toroidal boundary conditions. The leading magnetic exponents and the thermal exponent coincide with those of a multicritical point in the RSOS model with $p=6$ describing the continuous melting of a 6×1 commensurate phase to a disordered phase (Andrews *et al* 1984, Huse 1984). Furthermore, at the point D, defined by $\varepsilon = 2u/3\sqrt{3}$, $\delta = u/2$, $u \to \infty$, where the system decouples into the Ising and three-states Potts model, one has $c = 1.3$. Between B, S_1S_2 and D the central charge and the critical exponents change without a marginal operator to cause this effect. We find that the model has different types of second-order phase transitions, depending on the choice of coupling constants.

By successive combination of sectors of the model corresponding to the addition of characters of a given algebra we obtain representations of higher algebras classifying possible field theories. In B we find $N=1$ SUSY, by further combination $N=2$ SUSY and, finally, representations (5.5) of a new algebra which contains the $N=2$ superconformal algebra as a subalgebra. Along S_1S_2 the system is $N=1$ superconformal and ZF invariant. The connection of these two symmetries could be clarified in terms of character identities (4.32). The modular invariant periodic spectrum in the $c=1$ region coincides with that found by Ravanini and Yang (1987a) for $N=2$ supersymmetric models. Along the curve S_1S_2 it corresponds to one of the possible solutions for $N=1$ SUSY (Kastor 1987, Cappelli 1987) or for ZF symmetry respectively (Gepner and Qiu 1987). However, these models do not exhibit the full symmetry, since only a subspace of the corresponding Hilbert space is modular invariant.

Building up modular invariant partition functions with appropriately chosen Z_2 boundary conditions (Σ^3, C, $\Sigma^3 C$) for the Hamiltonian (1.1), all the remaining solutions of modular invariance for $N=1$ SUSY with $c=1$ (one more solution) and $c=\frac{5}{4}$ (two additional solutions) (Cappelli 1987) can be constructed. In the $c=1$ region we obtain the scalar-type partition function already found in the Ashkin-Teller model with periodic boundary conditions for a special value of the coupling constant (Baake *et al* 1987, Yang and Zheng 1987). Along curve S_1S_2 one obtains the other two solutions for $N=1$ SUSY with $c=\frac{5}{4}$. Whether there exist systems which are described by these partition functions is not yet known.

These remarks imply that the same critical exponents which appear in the $c=1$ region also appear in the Ashkin-Teller model since there also SUSY was found (Baake *et al* 1987, Yang and Zheng 1987) for a special value of the coupling constant. However, they do not belong to the same universality class. In fact, nowadays it has become clear that a universality class is not characterised by some critical exponents (apart from the other characteristics), but rather by specific sets of exponents, namely sets which lead to a modular invariant partition function of the system. The six-states quantum chain and the Ashkin-Teller model provide an illustration of this change in the concept of universality.

Acknowledgments

It is a pleasure to thank V Rittenberg for very fruitful discussions providing substantial support for the algebraic interpretation of the raw data. I also thank M Henkel for useful discussions particularly concerning the numerical work.

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